A primal-dual algorithm with Krylov subspace projection for computed tomography

AIP 2025

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joint work with Mirjeta Pasha (Virginia Tech) and Wim Dewulf

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KU Leuven

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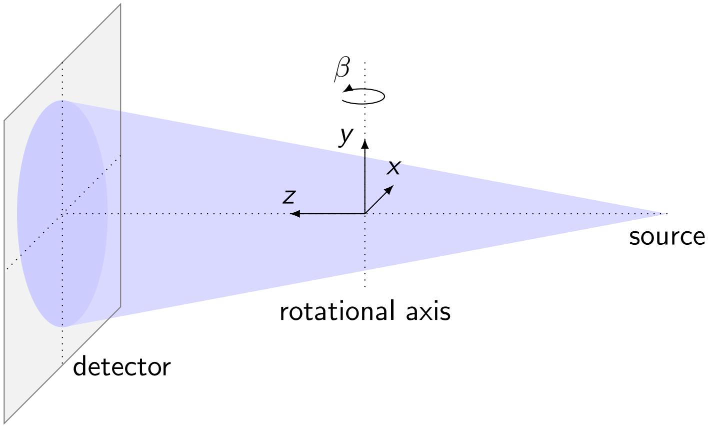
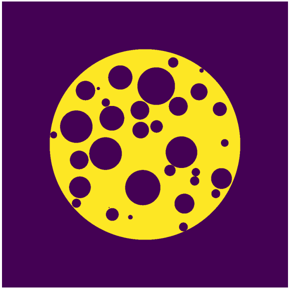


- Tomographic inverse problem:

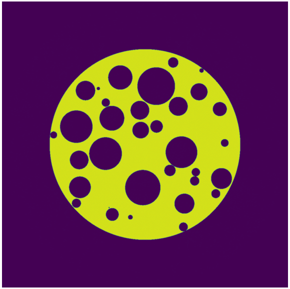
given data b and operator A, find u such that
$$\mathrm{A}u = b + \text{noise}$$

- First goal: Study *regularized* recostructions in cone-beam tomogphy

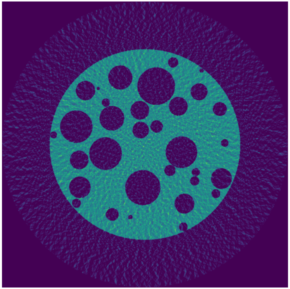
- Why regularized?: undersampled (few-view) data



Phantom



FDK 1024 views



FDK 60 views

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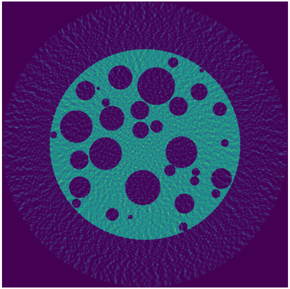
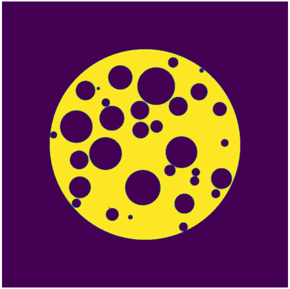


FDK 60 views



TV 60 views (Condat-Vu)

- We pose the reconstruction problem as

$$u^\dagger = \underset{u}{\operatorname{argmin}} \frac{1}{2}\|\mathrm{A}u - b\|^2 + \lambda\mathcal{R}(u), \quad \lambda_i > 0,$$

where $\mathcal{R}$ is a convex lower semicontinuos function and λ is a multivalued parameter.

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- and the parameter learning problem as

$$\begin{aligned} \lambda^{\dagger} = \quad & \underset{\lambda_i>0}{\operatorname{argmin}} \left\{ L(\lambda) := \tfrac{1}{2}\|u_\lambda - u_{\mathrm{true}}\|^2 \right\} \\ & \text{subject to } u_\lambda \in \underset{u}{\operatorname{argmin}} \, \tfrac{1}{2}\|\mathrm{A}u - b\|^2 + \lambda\mathcal{R}(u), \end{aligned}$$

where u_{true} is some available ground-truth image.

- For some $\mathcal{R}$, FISTA is a valid choise:

$$\begin{cases} u_{k+1} = \mathrm{prox}_{\gamma\lambda\mathcal{R}}\left(\hat{u}_k - \gamma \mathrm{A}^* \left(\mathrm{A}\hat{u}_k - b\right)\right) \\ t_{k+1} = \frac{1}{2}\left(1 + \sqrt{1 + 4t_k^2}\right) \\ \hat{u}_{k+1} = u_{k+1} + \dfrac{t_k - 1}{t_{k+1}}\left(u_{k+1} - u_k\right) \end{cases}$$

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- where computing the proximity operator

$$\operatorname{prox}_{\gamma\lambda\mathcal{R}}(b) = \underset{u}{\operatorname{argmin}}\left\{\frac{1}{2}\|u - b\|^2 + \gamma\lambda\mathcal{R}(u)\right\}$$

is now a *denosing* problem for a noisy image b regularized with $\mathcal{R}$.

- To avoid the denoising step, the Condat-Vu algorithm extends $\mathcal{R}$ as $\mathcal{R} \circ \mathsf{L} + h$ to now solve

$$\underset{u}{\operatorname{argmin}} \frac{1}{2}\|\mathsf{A}u - b\|^2 + \lambda\mathcal{R}(\mathsf{L}u) + \lambda h(u), \quad \lambda_i > 0,$$

with L a linear operator and h also convex and lower semicontinuos,

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- It iterates as

$$\begin{cases} u_{k+1} = \operatorname{prox}_{\gamma\lambda h}\left(u_k - \gamma \mathrm{L}^* w_k - \gamma \mathrm{A}^*\left(\mathrm{A}u_k - b\right)\right) \\ w_{k+1} = \operatorname{prox}_{\tau(\lambda\mathcal{R})^*}\left(w_k + \tau \mathrm{L} u_{k+1}\right) \end{cases}$$

- $\mathcal{R}$ could be *simple* now if we *split* it from L, the idea is to have a close-form expression of $\operatorname{prox}_{\mathcal{R}^*}(w_k)$

- Recall our learning problem

$$\lambda^\dagger = \underset{\lambda>0}{\operatorname{argmin}} \left\{ L(\lambda) := \frac{1}{2}\|u_\lambda - u_{\text{true}}\|^2 \right\}$$
$$\text{subject to } u_\lambda \in \underset{u}{\operatorname{argmin}} \frac{1}{2}\|\mathrm{A}u - b\|^2 + \lambda \mathcal{R}(u),$$

[1] De los Reyes, Schönlieb, Valkonen. J. Math. Imaging Vis. (2017)
[2] De los Reyes, Villacis. (2022)

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- We will solve it with a gradient-based algorithm, challenge: computing the gradient of L with respect to λ
- Appraches based on smoothing $\mathcal{R}$ and the adjoint state[1] exist, or based on nonsmooth trust regions[2]. However we will explore *automatic differentiation* (AD) approaches.
- The main challengue (with high-resolution problems) is now memory-related.

[1] De los Reyes, Schönlieb, Valkonen. J. Math. Imaging Vis. (2017)
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$$\begin{cases} u = \mathrm{prox}_{\gamma h}\left(u - \gamma \mathsf{L}^* w - \gamma \mathsf{A}^* \left(\mathsf{A} u - b\right)\right) \\ w = \mathrm{prox}_{\tau(\lambda \mathcal{R})^*}\left(w + \tau \mathsf{L} u\right) \end{cases}$$

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which has the form

$$\begin{cases} u = p\left(f(u, w)\right) \\ w = q\left(g(u, w), \lambda\right) \end{cases}$$

- Finally, we need the derivatives

$$\frac{\partial u}{\partial \lambda_i} = \frac{\partial p}{\partial f}\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial \lambda_i} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial \lambda_i}\right)$$

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- Mind the terms $\left\{\frac{\partial q}{\partial g}, \frac{\partial q}{\partial \lambda_i}\right\}$

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- Mind the terms $\left\{\frac{\partial q}{\partial g}, \frac{\partial q}{\partial \lambda_i}\right\}$
- Stored RAM memory in one GP / CV / informed CV iteration

	GP	CV	iCV
stored memory (MB.)	153	82	25

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- If needed, truncated backpropagation could be employed as well

- Example: total variation and nonnegativity :

$$\mathcal{R} = \| \cdot \|_{2,1}, \quad L = \nabla, \quad h = \imath_{\mathbb{R}_+}$$

$$q(w, \lambda) = \text{prox}_{\lambda \mathcal{R}^*}(w) = \Pi_{\|w\|_{2,\infty} \leq \lambda}(w) = \frac{w}{\max\left\{1, \lambda^{-1}|w|_2\right\}}$$

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then we have the derivatives

$$\frac{\partial}{\partial\lambda} q(w,\lambda) = \frac{w|w|_2}{\lambda^2 \max^2\left\{1, \lambda^{-1}|w|_2\right\}} H(\lambda^{-1}|w|_2)),$$

$$\frac{\partial}{\partial w} q(w,\lambda) = \frac{H(\lambda^{-1}|w|_2)w^2/|w|_2}{\lambda \max^2\left\{1, \lambda^{-1}|w|_2\right\}} - \frac{1}{\max^2\left\{1, \lambda^{-1}|w|_2\right\}}$$

with H the Heaviside step function

- Example: sparsity on some new basis and nonnegativity :

$$\mathcal{R} = \|\cdot\|_1, \quad L = \mathcal{W}, \quad h = \imath_{\mathbb{R}_+}$$

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- Example: total *generalized* variation and nonnegativity :

$$
\begin{aligned}
& \underset{u}{\operatorname{argmin}} \frac{1}{2}\|\mathsf{A}u - b\|^2 + \lambda \mathcal{R}(\mathsf{L}(u,v)) + \lambda h(u,v), \quad \lambda_i > 0, \\
= \; & \underset{u,v}{\operatorname{argmin}} \frac{1}{2}\|\mathsf{A}u - b\|^2 + \lambda_1 \|\nabla u - v\|_{2,1} + \lambda_2 \|Ev\|_{2,1} + h(u,v)
\end{aligned}
$$

and we can still apply Condat-Vu with

$$
\mathsf{L} = \begin{pmatrix} \nabla & -Id \\ 0 & E \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} \|\cdot\|_{2,1} \\ \|\cdot\|_{2,1} \end{pmatrix}, \quad h = \begin{pmatrix} \imath_{\mathbb{R}_+} \\ 0 \end{pmatrix}
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then we know from tv the derivatives

$$\frac{\partial}{\partial\lambda}\mathrm{prox}_{\lambda\mathcal{R}^*}(w), \quad \frac{\partial}{\partial w}\mathrm{prox}_{\lambda\mathcal{R}^*}(w)$$

with now $w = (w_1, w_2)$ the dual variable of (u, v)

- Example: *infimal convolution* total variation and nonnegativity :

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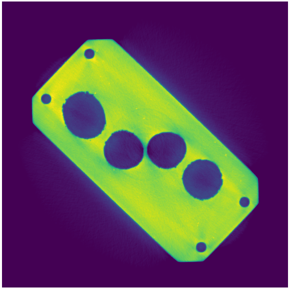
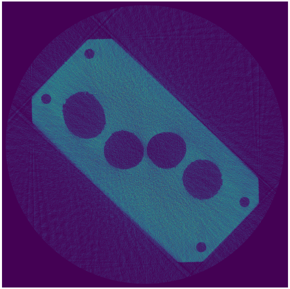
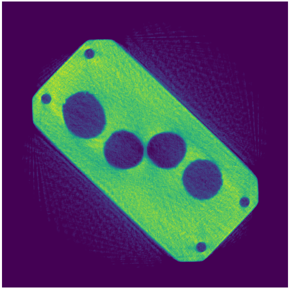
- PG, Bellens, Dewulf, arXiv:2412.10034 (2025)



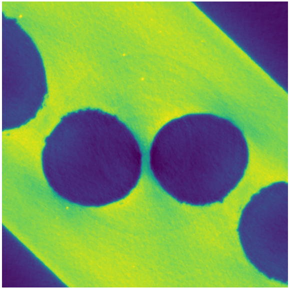
3000 projections



100 projections FDK



100 projections LSQR



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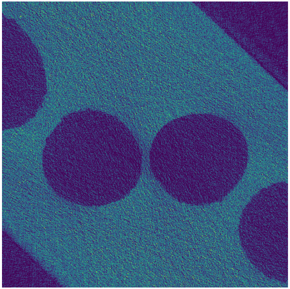

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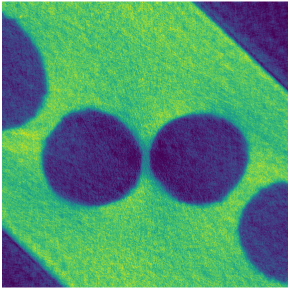

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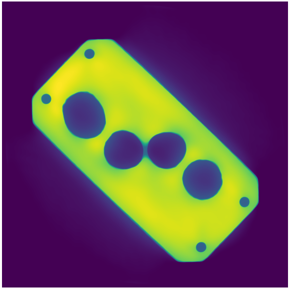
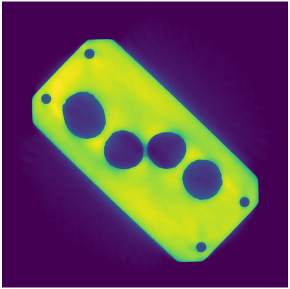
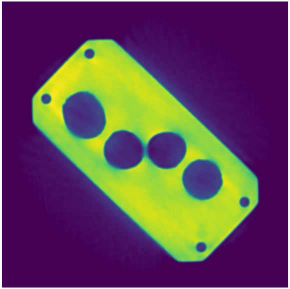


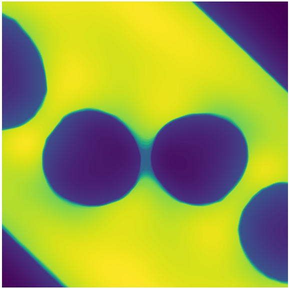
tv

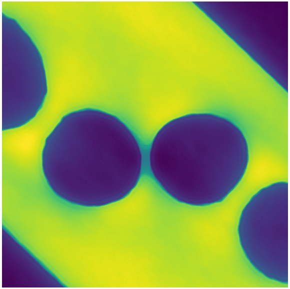


ictv



tgv


tv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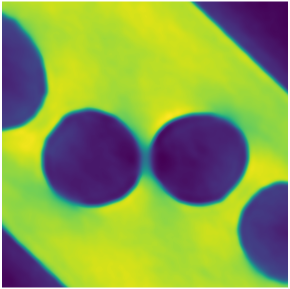
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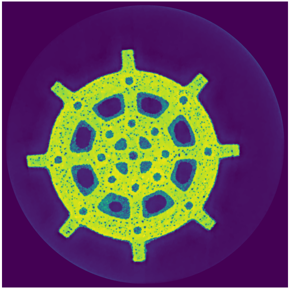

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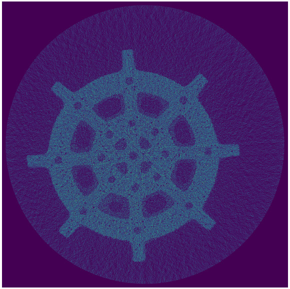

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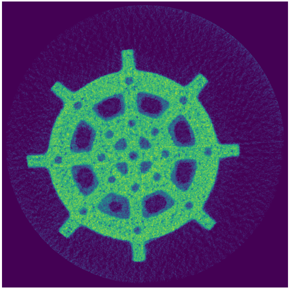

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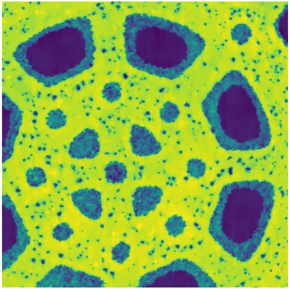

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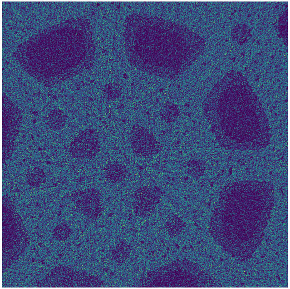

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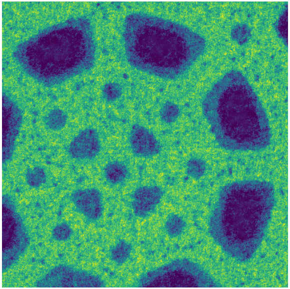
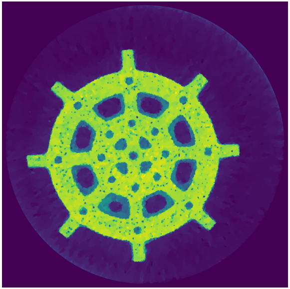
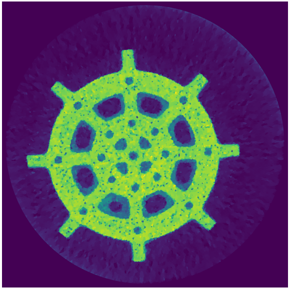


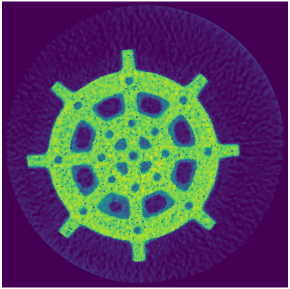
100 projections LSQR



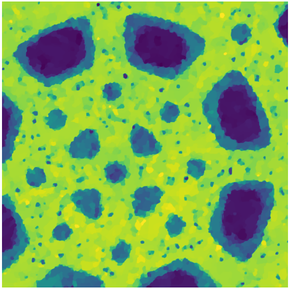
tv



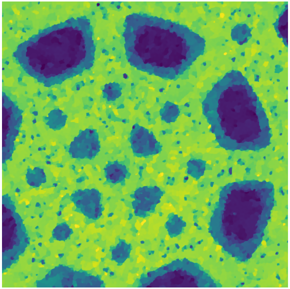
ictv



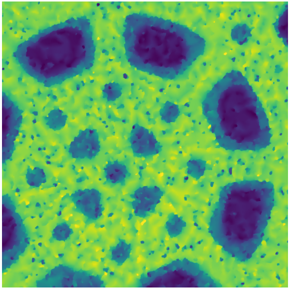
tgv



tv



ictv



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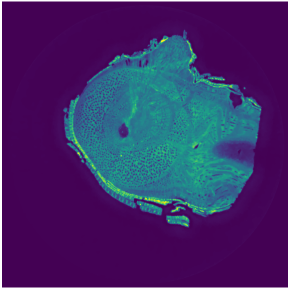
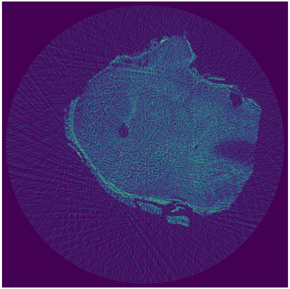
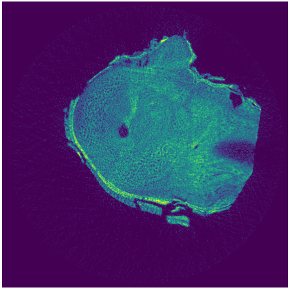


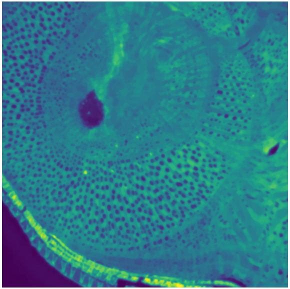
3000 projections



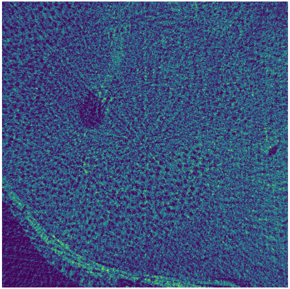
100 projections FDK



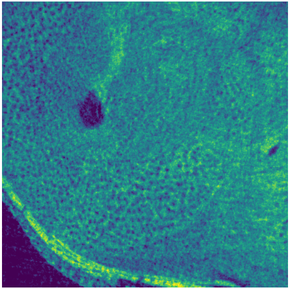
100 projections LSQR



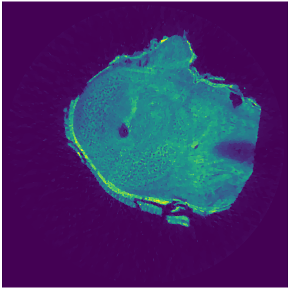
3000 projections



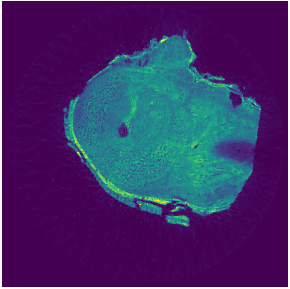
100 projections FDK



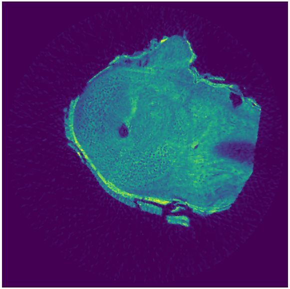
100 projections LSQR



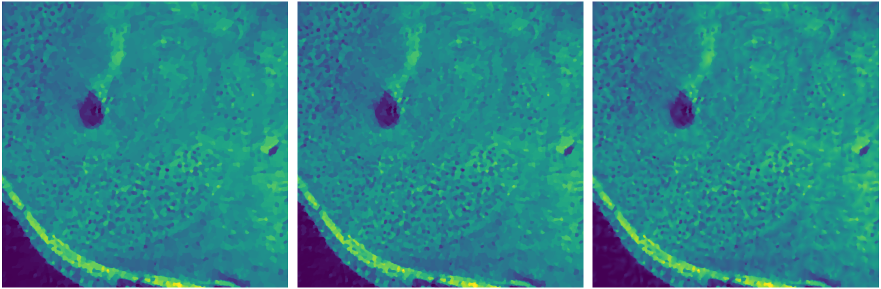
tv



ictv



tgv



tv ictv tgv

- Recall our reconstruction problem

$$\underset{u}{\operatorname{argmin}} \frac{1}{2}\|\mathsf{A}u - b\|^2 + \lambda\mathcal{R}(\mathsf{L}u) + h(u), \quad \lambda_i > 0,$$

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$$\operatorname*{argmin}_{u} \tfrac{1}{2}\|\mathsf{A}u - b\|^2 + \lambda\mathcal{R}(\mathsf{L}u) + h(u), \quad \lambda_i > 0,$$

- A can be reduced by performing l ($\ll \mathrm{pixels}$) steps of Golub-Kahan bidiagonalization giving the decomposition

$$\mathsf{AV} = \mathsf{UB},$$

where the orthogonal matrix $\mathsf{V} \in \mathbb{R}^{n\times\ell}$ spans the Krylov subspace

$$K_\ell(\mathsf{A}^\mathsf{T}\mathsf{A}, \mathsf{A}^\mathsf{T}b) = \mathrm{span}\{\mathsf{A}^\mathsf{T}b, (\mathsf{A}^\mathsf{T}\mathsf{A})\mathsf{A}^\mathsf{T}b, \ldots, (\mathsf{A}^\mathsf{T}\mathsf{A})^{\ell-1}\mathsf{A}^\mathsf{T}b\},$$

then we solve

$$\min_{y\in\mathbb{R}^\ell} \tfrac{1}{2}\|\mathsf{UB}y - b\|^2 + \mathcal{R}\left(\mathsf{LV}y\right) + h\left(\mathsf{V}y\right),$$

and because of $\mathsf{U}^\mathsf{T} b = \|b\| \mathsf{e}_1$ with $\mathsf{e}_1 = [1, 0, ..., 0]^\mathsf{T}$, we are solving

$$\min_{y \in \mathbb{R}^\ell} \frac{1}{2} \|\mathsf{B} y - \|b\| \mathsf{e}_1\|^2 + \mathcal{R}(\mathsf{LV} y) + h(\mathsf{V} y) \tag{1}$$

which can already be solved with Condat-Vu

$$\begin{cases} y_{k+1} = \mathrm{prox}_{\gamma h \circ \mathsf{V}} \left(y_k - \gamma \mathsf{V}^\mathsf{T} \mathsf{L}^\mathsf{T} w_k - \gamma \mathsf{B}^\mathsf{T} (\mathsf{B} y_k - \beta \mathsf{e}_1) \right) \\ w_{k+1} = \mathrm{prox}_{\tau (\lambda \mathcal{R})^*} (w_k + \tau \mathsf{LV} y_{k+1}) \end{cases}$$

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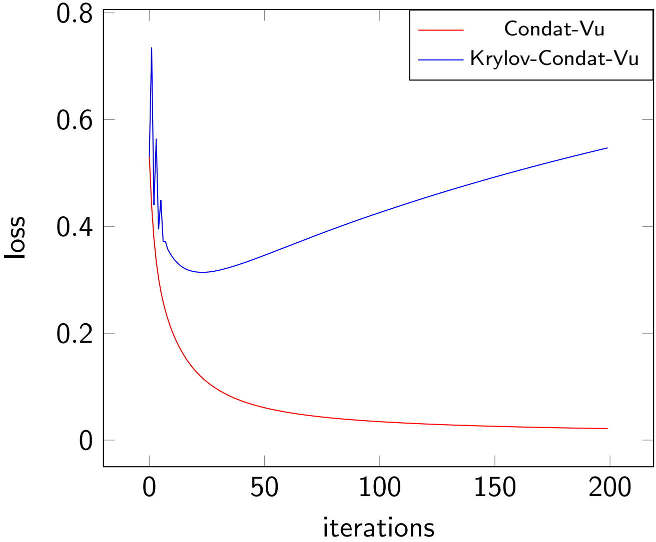
we also have $\operatorname{prox}_{\gamma h \circ \mathsf{V}} = \mathsf{V}^\mathsf{T} \circ \operatorname{prox}_{\gamma h} \circ \mathsf{V}$, the iteration becomes

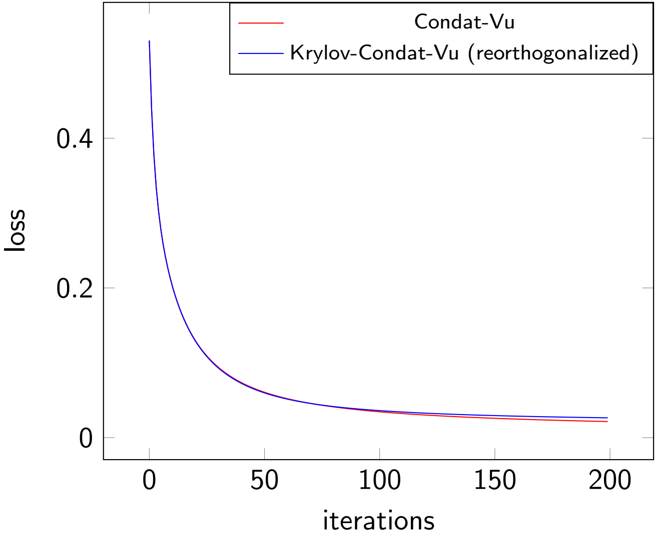
$$\begin{cases} y_{k+1} = \mathsf{V}^\mathsf{T} \operatorname{prox}_{\gamma h} \left(-\gamma \mathsf{L}^\mathsf{T} w_k + \mathsf{V} \left(y_k - \gamma \mathsf{B}^\mathsf{T} (\mathsf{B} y_k - \beta \mathsf{e}_1) \right) \right) \\ w_{k+1} = \operatorname{prox}_{\tau (\lambda \mathcal{R})^*} \left(w_k + \tau \mathsf{LV} y_{k+1} \right) \end{cases}$$

then (y_k) convergences weakly to a solution y of (1) and $x = \mathsf{V} y$ is our final reconstruction.

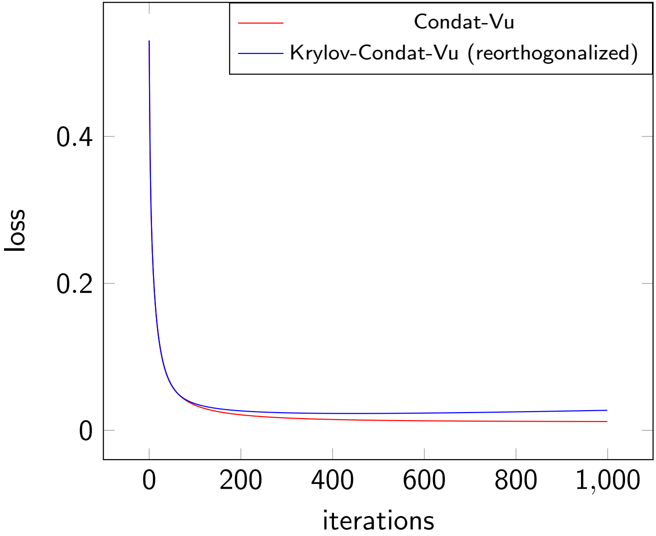
Convergence behaviour

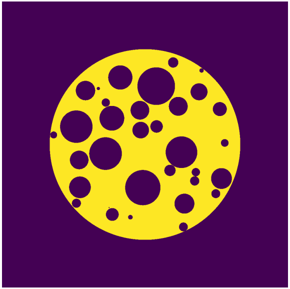


Convergence behavior

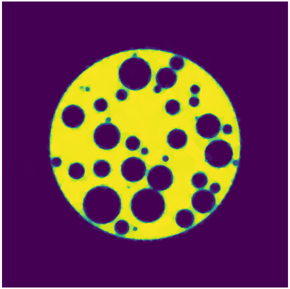


Convergence behavior

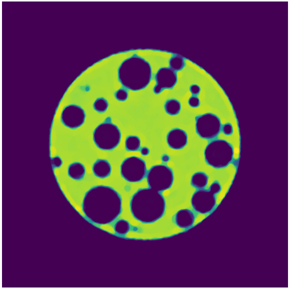




sample



Condat-Vu 60 projections



Krylov-Condat-Vu 60 projections

- Automatic differentiation in bilevel learning can be feasible even in high dimensional problems.
- Condat-Vu acceleration seems promising as well[3]

[3]Driggs, Ehrhardt, Schönlieb, Tang. 2024

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- Automatic differentiation in bilevel learning can be feasible even in high dimensional problems.
- Condat-Vu acceleration seems promising as well[3]
- Krylov-Condat-Vu could accelerate considerably the computing time in this kind og problems (particularly in the learning step)
- Still needs a better understanding around numerical instabilities

[3] Driggs, Ehrhardt, Schönlieb, Tang. 2024

$\rightarrow$ thanks !